CS 237: Probability in Computing

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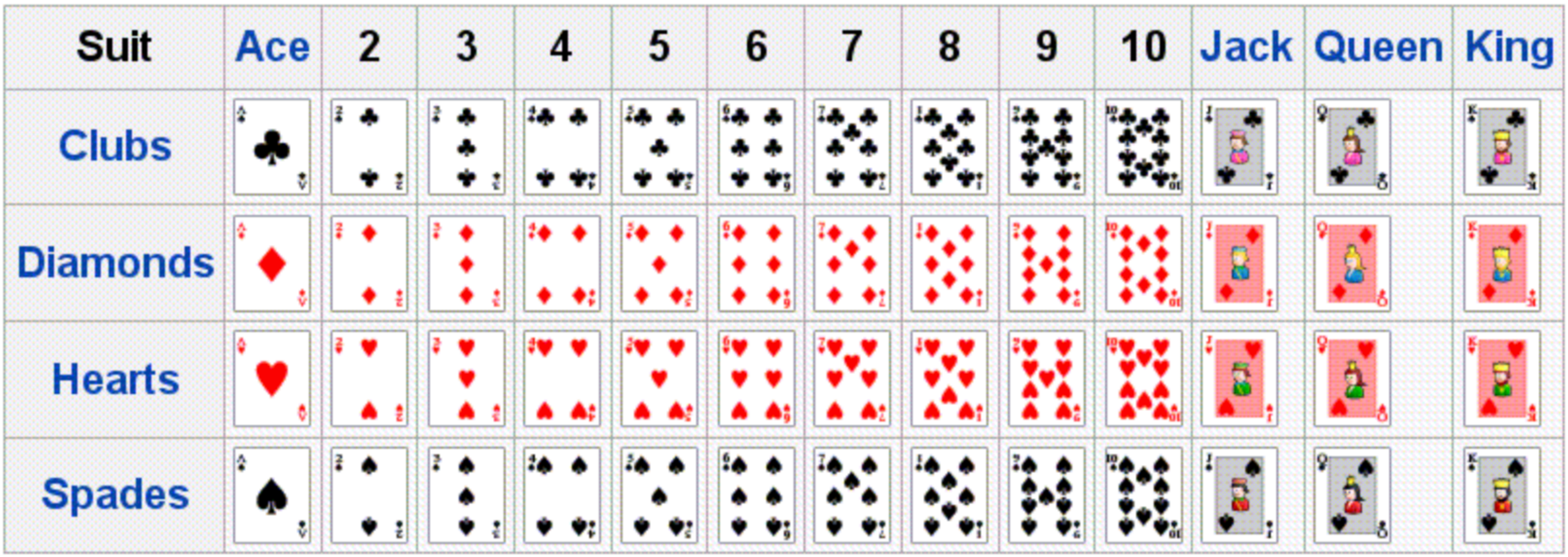
Lecture 8:

- Poker Probability (analytical results for lab problems in HW 05)
- Using combinations to count partitions;
- Ordered partitions vs. unordered partitions;
- Overcounting due to duplicates among subsets.

Finite Combinatorics and Probabilities

Poker Probabilities

Example set of 52 poker playing cards



A poker hand is 5 cards (a set) chosen without replacement.

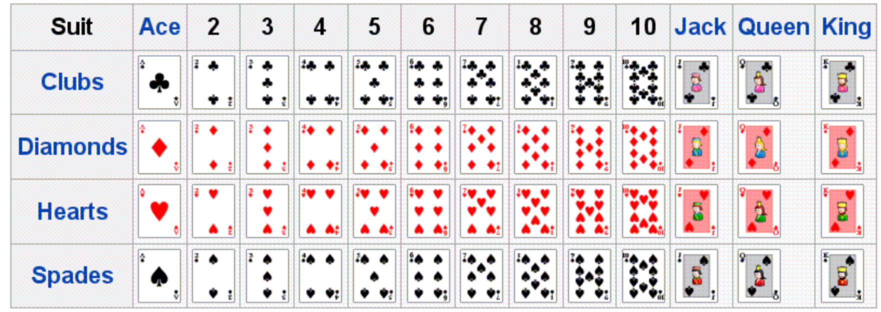
How many possible poker hands?

$$|S| = \binom{52}{5} = 2,598,960$$

Poker Probability

Poker hands are a great example of how to think about probability involving sets.

Example set of 52 poker playing cards



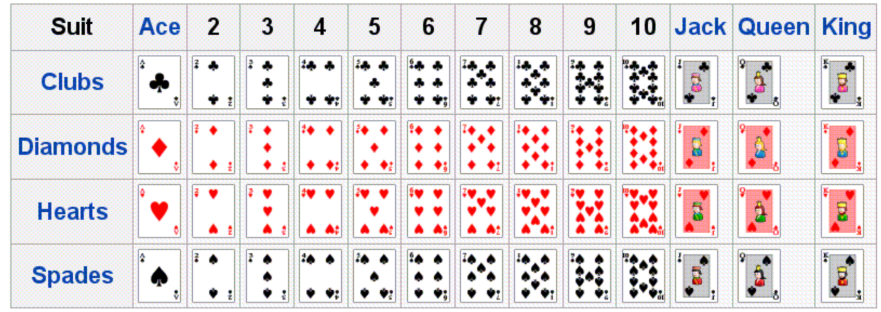
Example: What is the probability that you get a hand with 3 red cards and 2 black cards?

(Hint: Construct the hand by calculating how many such hands are possible, by constructing independent parts of the hand, and multiplying....)

Poker Probability

Poker hands are a great example of how to think about probability involving sets.

Example set of 52 poker playing cards

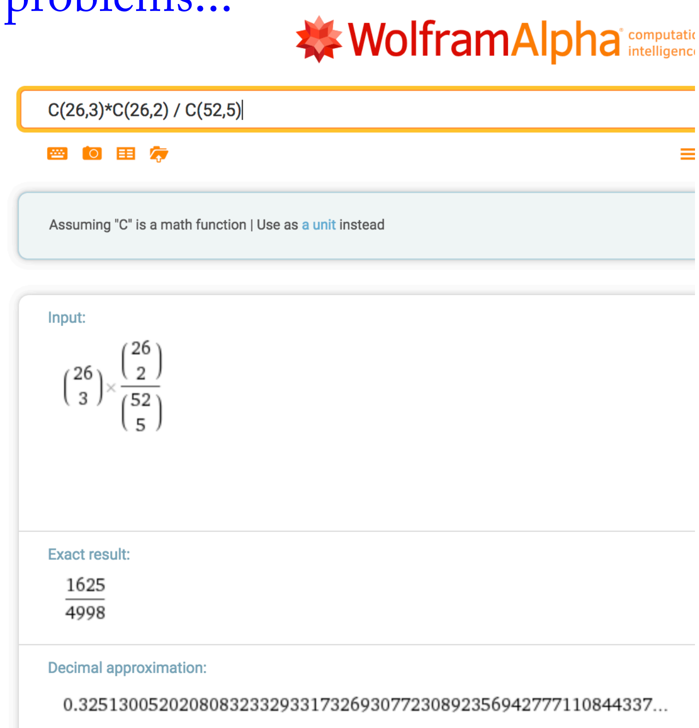


Example: What is the probability that you get a hand with 3 red cards and 2 black cards?

Solution:

$$\frac{\binom{26}{3}\binom{26}{2}}{\binom{52}{5}} = 0.3251$$

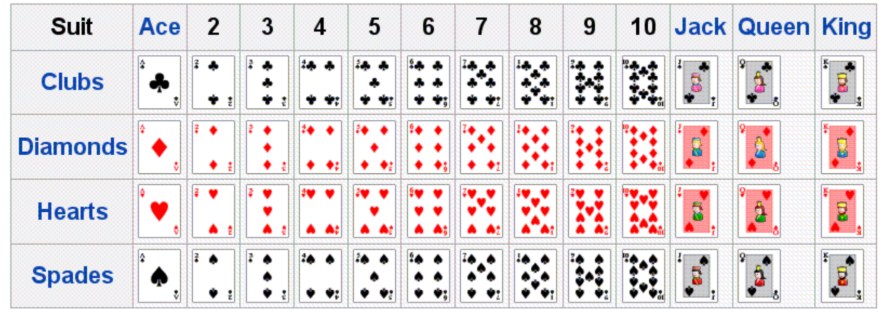
By the way, Wolfram Alpha is the way to go when doing these problems...



Poker Probability

Poker hands are a great example of how to think about probability involving sets.

Example set of 52 poker playing cards



Example: What is the probability that you get a hand with 3 red cards and 2 black cards?

$$\frac{\binom{26}{3}\binom{26}{2}}{\binom{52}{5}} = 0.3251$$

Digression: Notice that you can also calculate this using **sequences** and **permutations**, but it is a bit more complicated, and you have more opportunities to get something wrong...

What is the probability of the exact sequence R R R B B ?

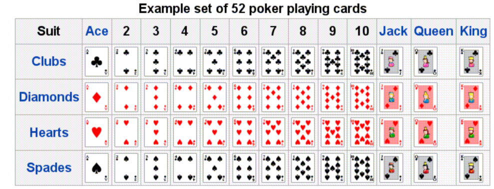
$$\frac{26}{52} * \frac{25}{51} * \frac{24}{50} * \frac{26}{49} * \frac{25}{48} = 0.0325$$

Now unorder it! How many permutations of this sequence of 5 symbols with duplicates?

$$\frac{5!}{3!\,2!} = \frac{120}{6 * 2} = 10$$

0.03251 * 10 = 0.3251

Example set of 52 poker playing cards



Poker Probability

Problem: What is the probability that a five-card hand has at least 3 Diamonds?

Solution: You need to separate this problem into cases, and might as well choose 3, 4, or 5 Diamonds, and for each find the probability and sum:

$$P(\text{ 3 Diamonds }) = \frac{\binom{13}{3}\binom{39}{2}}{\binom{52}{5}} = 0.0815$$

$$P(\text{ 4 Diamonds }) = \frac{\binom{13}{4}\binom{39}{1}}{\binom{52}{5}} = 0.0107$$

$$P(\text{ 5 Diamonds }) = \frac{\binom{13}{5}}{\binom{52}{5}} = 0.0005$$

These sum to 0.0928.

Poker Probability

Problem: What is the probability of a Flush (all the same suit)?

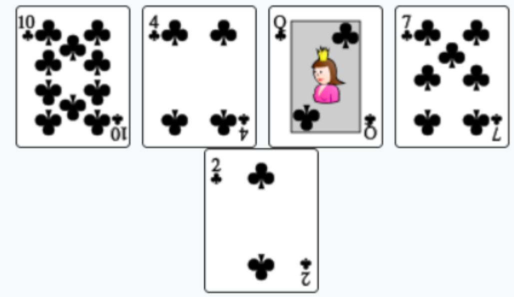
Solution: Choose a suit and then choose 5 cards from that suit:

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} = 0.00198079$$

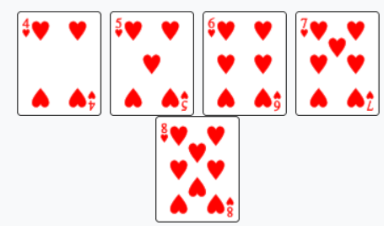
Note: This is the **cumulative** probability, in that it is a flush, but includes the straight and royal flushes. If we wish to exclude them, we must subtract all 40 of them:

$$\frac{\binom{4}{1}\binom{13}{5} - \binom{10}{1}\binom{4}{1}}{\binom{52}{5}} = 0.00196540$$

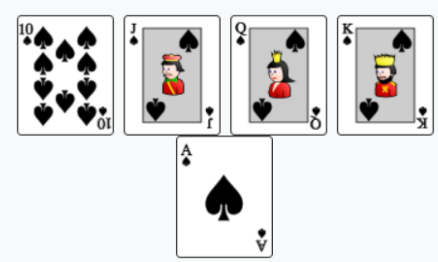
Flush (excluding royal flush and straight flush)



Straight flush (excluding royal flush)



Royal flush



Poker Probability

Problem: What is the probability of a Straight? Assume that Ace can be below 2 or above King.

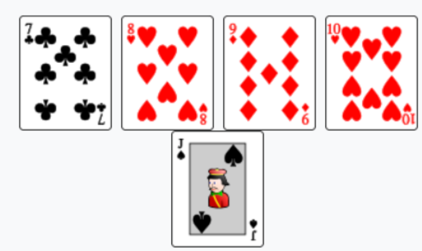
Solution: There are 10 sequences which form a straight, so just choose one of the 10 and then suits for each of the 5 cards:

A	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8
5	6	7	8	9
6	7	8	9	10
7	8	9	10	J
8	9	10	J	Q
9	10	J	Q	K
10	J	Q	K	A

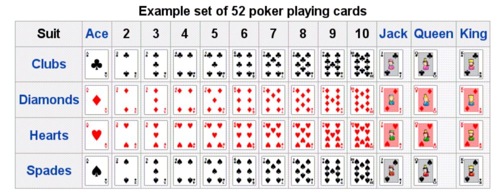
$$\frac{\binom{10}{1} * \binom{4}{1}^5}{\binom{52}{5}} = 0.00394$$

$$\frac{\binom{10}{1} * \binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}}{\binom{52}{5}} = 0.003925$$

Straight (excluding royal flush and straight flush)

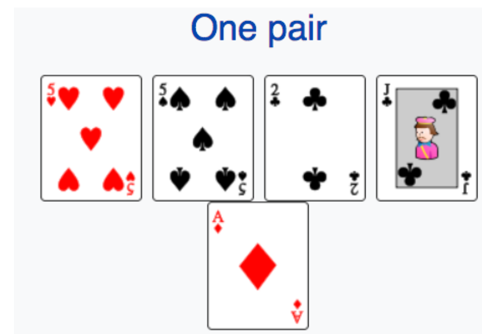


Poker Probability



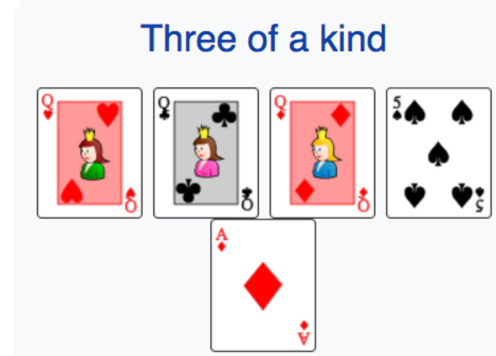
Problem: What is the probability of a Pair, 3-of-a-Kind, and 4-of-a-Kind?

Solution: First choose the denomination of the 2, 3 or 4 of a kind, then the suits of those cards, then the remaining cards of different denominations, again chosing the denomination, then the suits::



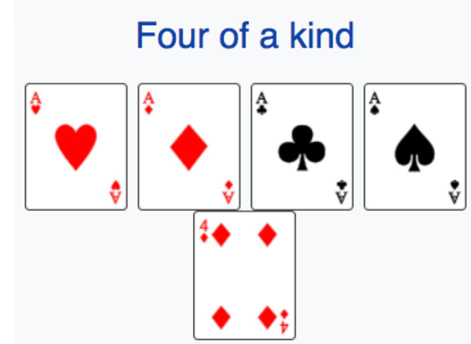
Pair:

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{5}} = 0.4226$$



3-of-a-Kind:

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}} = 0.0211$$



4-of-a-Kind:

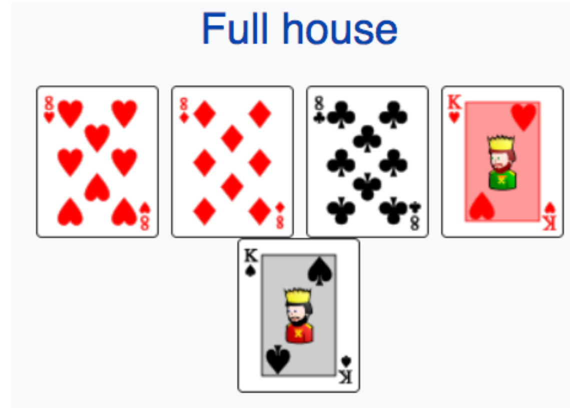
$$\frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}} = 0.00024$$

Problem: Suppose in our class we have 160 students with 90 men and 70 women. I want to choose teams for an in-class demo with 5 men, 5 women, and also a scorekeeper (who can be anyone not on a team, man or woman). How many ways can I choose the teams and scorekeeper?

Solution: First choose the 5 men from the 90, then the 5 women from the 70, then one scorekeeper from the 150 people not on teams:

$$\binom{90}{5} * \binom{70}{5} * \binom{150}{1} = 79,787,790,884,062,800$$

Poker Probability



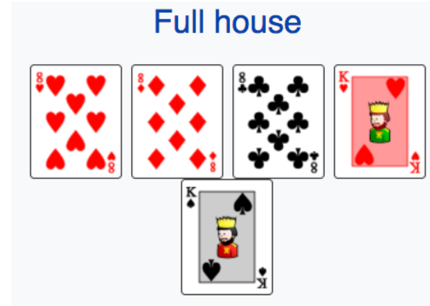
Problem: What is the probability of a Full House (3 of one denomination and 2 of another)?

Solution: First choose the denomination of the 3, then those 3 suits, then the denomination of the 2, then those 2 suits:

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = 0.00144$$

We will consider one more hand, Two Pair, after some consideration of partitions.

Poker Probability



Question: Why is this not "choose the two ranks, then the suits for each"?

$$\frac{\binom{13}{2}\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = 0.00144$$

It's the difference between sequences and sets!

Number of sets of 2 from 13: C(13, 2):

Number of sequences of 2 from 13, P(13, 2):

$$\binom{13}{2} = 78 \qquad P(13, 2) = 13 \cdot 12 = 156$$

Counting Sets: Power Set

The Power Set of a set S is the set of all subsets (= set of all events):

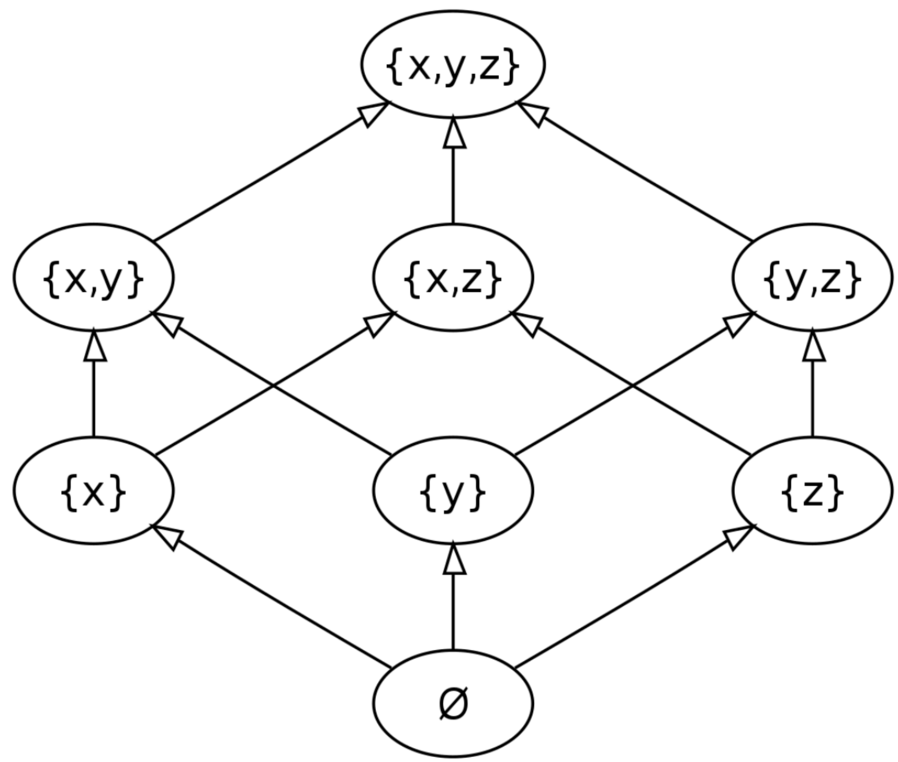
$$\mathcal{P}(S) \quad =_{\text{def}} \quad \{ A \mid A \subseteq S \}$$

the cardinality of Power Set: $|\mathcal{P}(S)| = 2^{|S|}$

This is easy to see if we consider the **enumeration** of all sequences of { T, F } of length |S|, stating which elements of S are in the subset:

```
{ x, y }

x  y  z
T  T  F
```

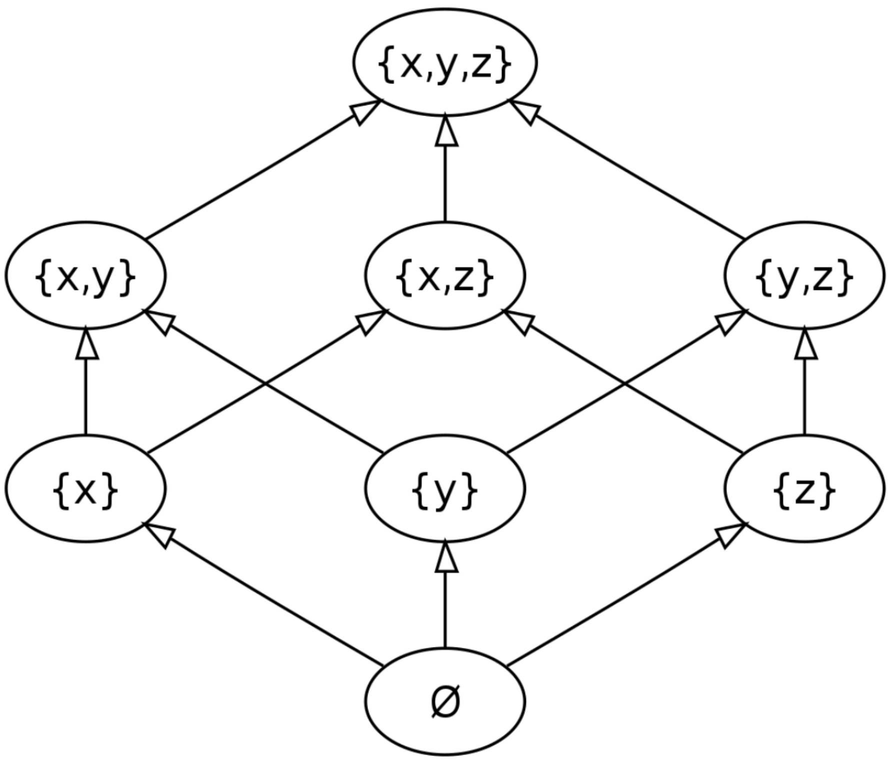
$|\{T,F\}|^{|S|} = 2^{|S|}$



Counting Sets: Power Set and Combinations

There is of course a strong connection between the power set and combinations:

C(N, K) = how many subsets of size K from a set of size N.

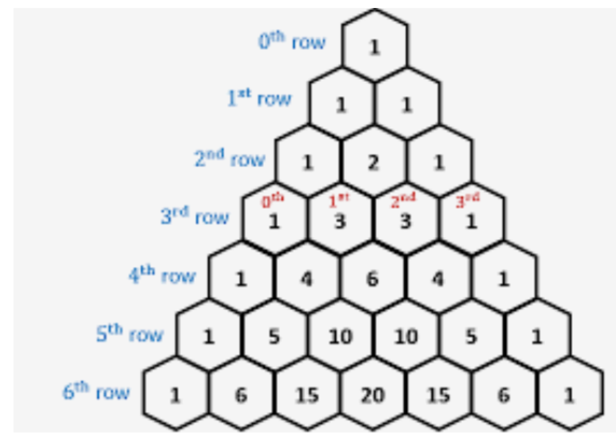


$$\binom{3}{3} = 1$$

$$\binom{3}{2} = 3$$

$$\binom{3}{1} = 3$$

$$\binom{3}{0} = 1$$



$$|\mathcal{P}(S)| = 2^{|S|} = \sum_{k=0}^{|S|} \binom{|S|}{k}$$

Counting Sets: Power Set and Combinations

Problem: A pizza shop claims they serve "more than 1000 kinds of pizza." You investigate and find they offer 10 different toppings (including cheese and tomato sauce among the 10). Is their claim correct? What about if we insist that a pizza must have cheese and tomato sauce at the very least?

Solution: Technically, yes, if you include all possible combinations of toppings, including cheese or no cheese and tomato sauce or no tomato sauce:

$$2^{10} = 1024$$

But this is a little funny, as it includes the empty set (no toppings, just bare crust!).

If you insist that "pizza" must have cheese and tomato sauce, then we have only

$$2^{8} = 256$$

Counting Sets: Power Set and Combinations

Problem: A pizza shop claims they serve "more than 1000 kinds of pizza." You investigate and find they offer 10 different toppings (including cheese and tomato sauce among the 10). Is their claim correct? What about if we insist that a pizza must have cheese and tomato sauce at the very least?

Solution:

Counting Sets: Partitions

A partition of a set S is a set of disjoint subsets which include every member of S:

S = { 1, 2,3,4,5,6,7,8,9,10 }

Partitions or not ?

{ {1, 2, 3, 4}, {5, 6, 7, 8, 9, 10} }

{ {1}, {2, 4}, {3, 6, 8}, {5, 7, 9}, {10} }

{ {1, 2, 3, 4}, {5, 6, 7, 8, 9} }

{ {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}, {10} }

{ {1, 2, 3, 4, 5}, {5, 6, 7, 8, 9, 10} }

Counting Sets: Partitions

Counting partitions: always a good idea to try some examples first ...

Problem. Suppose we have five students { A, B, C, D, E }

We want to divide them into two teams of 3 and 2 people each. How many ways can we do this?

$$\binom{5}{3} = 10$$

Team of 3	Team of 2
A B C	D E
A B D	C E
A B E	C D
A C D	B E
A C E	B D
A D E	B C
B C D	B E
B C E	A D
B D E	A C
C D E	A B

Note: Once we have chosen the team of 3, the other team is determined!

For each one of these, there is only one set of 3 and set of 2, e.g.,

{ {A,B,C}, {D,E} }

Counting Sets: Partitions

Now let's try 2 teams of 2:

Problem. Suppose we have four students { A, B, C, D }

We want to divide them into two teams of 2 people each. How many ways can we do this? Is it this?

$$\binom{4}{2} = 6$$

Team of 2	Team of 2
A B	C D
A C	B D
A D	B C
B C	A D
B D	A C
C D	A B

Counting Sets: Partitions

Now let's try 2 teams of 2:

Problem. Suppose we have four students { A, B, C, D }

We want to divide them into two teams of 2 people each. How many ways can we do this?

$$\binom{4}{2} = 6$$

Not correct! We have overcounted by a factor of 2.

Team of 2	Team of 2
A B	C D
A C	B D
A D	B C
B C	A D
B D	A C
C D	A B

Same WAY of dividing into 2 teams!

As sets, these are the same way of building a partition:

{ {A,B}, {C,D} } is same set of sets as:

{ {C,D}, {A,B} }

Counting Sets: Partitions

Now let's try 3 teams of 2:

Problem. Suppose we have six students { A, B, C, D, E, F }

We want to divide them into 3 teams of 2 people each. How many ways can we do this?

$$\binom{6}{2}\binom{4}{2} = 15 * 6 = 90$$

Team of 2		Team of 2	Team of 2
A B		C D	E F
	....		
A B		E F	C D
	....		
C D		A B	E F
	....		
C D		E F	A B
	....		
E F		A B	C D
	....		
E F		C D	A B

Note: Once we have chosen the first 2 teams of 2, the last team is determined!

All these "ways" of dividing into 3 teams of equal size are the same!

Overcounting by P(3,3) = 3!, correct answer is:

$$\frac{\binom{6}{2}\binom{4}{2}}{3!} = \frac{90}{6} = 15$$

The **Unordering Principle** strikes again!

Counting Sets: Partitions

Problem. Suppose we have 15 students and want to divide them into

2 teams of 3,
4 teams of 2, and
a single student who will be referee.

How many ways of doing this are there?

Solution: Use permutations with duplicates to remove the duplicates among teams you can't distinguish by size:

$$\frac{\binom{15}{3}\binom{12}{3}\binom{9}{2}\binom{7}{2}\binom{5}{2}\binom{3}{2}}{2! * 4!} = \frac{2,270,268,000}{48} = 47,297,250$$

Counting Sets: Partitions

Now suppose we distinguish the teams by NAME.

Problem. Suppose we have four students { A, B, C, D }

We want to divide them into two teams of 2 people each called ”Attackers” and “Defenders.” How many ways can we do this?

$$\binom{4}{2} = 6$$

Attackers	Defenders
A B	C D
A C	B D
A D	B C
B C	A D
B D	A C
C D	A B

Now there is no overcounting!
Switching attackers and defenders gives you a different **way**.
There are no **duplicate** ways.

This may seem obscure, but think about experiments involving a "test group" (who take a new drug) and a "control group" (who take a placebo). Switching the groups makes a difference!

Counting Sets: Partitions

Problem. Suppose we have 15 students and want to divide them into

-- 2 teams of 3, named “MIT Attackers” and “Harvard Attackers”
-- 4 teams of 2, all defenders (all unnamed); and
-- a single student who will be referee.

How many ways of doing this are there?

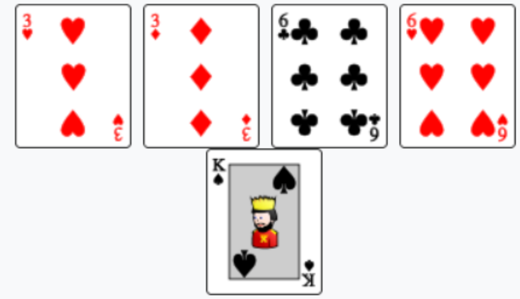
Solution: Use multinomial coefficients to remove the duplicates among teams you can’t distinguish by size or name:

$$\frac{\binom{15}{3}\binom{12}{3}\binom{9}{2}\binom{7}{2}\binom{5}{2}\binom{3}{2}}{4!} = \frac{2,270,268,000}{24} = 94,594,500$$

Poker Probability -- One Last Time:

Problem: What is the probability of Two Pair (2 of one denomination and 2 of different denominations)?

Two pair



Solution: First choose the denomination of the first pair, then those 2 suits, then the denomination of the second pair, then those 2 suits, then the remaining card:

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} = 0.09508$$

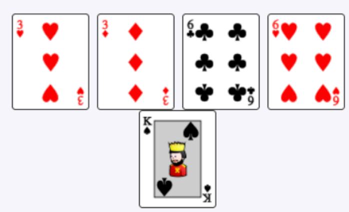
But wait.... This doesn't correspond to the web page OR our experiments, which seem to suggest it is too high by a factor of 2. What is wrong?

Problem 5 (C): What is probability of Two Pairs in Poker?

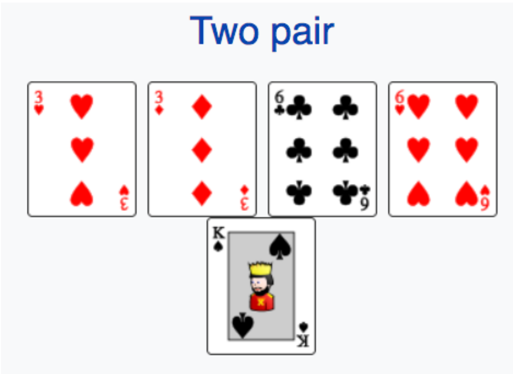
In [98]:

```
# probability should be close to analytical value of 0.047539

seed(0)
num_trials = 10**5

print('Probability of two pairs in poker is ' + str(probability_of
```

```
Probability of two pairs in poker is 0.04618
```

Two pair 	858	123,552	4.7539%

Poker Probability One Last Time:



Just another example of you-know-what, in this case, overcounting the two pairs:

2D	2H	3C	3D	5S
3C	3D	2D	2H	5S

These are the same hand, but would be counted twice!

Same problem as: you have 52 students, and want to select 2 teams of 2, plus a referee.

So we could divide by 2! to get the right number:

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}}{2 * \binom{52}{5}} = 0.04754$$

OR we could choose a set of 2 ranks to get the two pairs:

$$\frac{\binom{13}{2}\binom{4}{2}^2\binom{44}{1}}{\binom{52}{5}}$$